Geometric Strategies in Creating Origami Paper Lampshades: Folding Miura-ori, Yoshimura, and Waterbomb Tessellations

Abstract

This chapter describes geometric strategies in origami design to create paper lanterns that gleam with the luminous gradations of light. While the design of origami paper lampshades is largely based on origami design, it also presents new challenges due to its specific set of design constraints as a new genre of functional art. This chapter intends to address this specific set of the design constraints through understanding the underlying mathematics in origami design and provide a set of tools for constructing origami lampshade that results in high aesthetic quality. It begins with an introduction to origami design and its relationship to mathematics and the historical background on origami paper lanterns, it then discusses various geometric strategies for creating origami paper lampshades based on the Miura-ori, Yoshimura, and Waterbomb tessellations. The emphasis is on specific mathematical requirements for creating functional light art with dramatic and perceptual effects of translucent light.

Keywords

Origami, Lampshade, geometry, tessellation, flat-foldable

1. Introduction

The original purpose of origami was to create various shapes, ranging from animal figures to objects, and as decorative items to be used in religious and ceremonial activities (pbs.org 2017). However, the craft techniques were also used to create geometrically abstract and non-representational functional objects. Perhaps one of the earliest functional models of origami was a folded gift box called Tamatebako. or "magic treasure chest." This first appeared in a Japanese book published in 1743 called Ranma Zushiki which documented Edo-period design (Kasahara 2004). Outside of Japanese origami tradition, there are many disparate areas of functional folding endeavours, ranging from the 15th century European napkin-folding(Sallas 2010) to early 20th century Bauhaus architecture (Wingler and Stein 1976). Today, constructing three-dimensional surfaces from two-dimensional sheet material has inspired artists, designers, architects, and engineers to come up with folded sculptural forms that react to kinetic movements (Wu, Ressel, and Overton 2018) as well as the interplay of light and shadow in fashion (Morriseey 2019), products (Wu 2018), and architecture (Choma 2018). Many of these new designs relay on the techniques in computational geometry and mathematics.

Unfolding a piece of origami reveals the intricate crease patterns that define the geometrical transformation of folding the piece of material. The lines of the crease patterns will keep their length constant during the paper-folding transformation and that paper cannot intersect itself. Mathematicians call these geometric transformations isometry and injection (Lang 2018). These remarkable properties found in paper folding suggests that there are deep connections between mathematics and origami. As early as the mid-to-late 1890s, the geometry of origami was studied (Sundara Rao 1901). In 1930s, Margherita P. Beloch proved that instead of using straightedges and compasses, origami can be used to solve cubic equations (Hull 2011). While mathematical studies of origami go back at least as far as 1890s or 1930s, a new research field, origami mathematics, has been developed in the last decades in order to understand the mathematical formalization underlying paper folds. Beginning in the 1960s, computer scientists Ron Resch designed and folded paper forms using mathematical and computational algorithms (Resch 1968, Resch 1973). Another computer scientist, David Huffman, developed many new mathematical concepts on folding based on Resch's work in the 1970s (Huffman 1976). In 1997, mathematician Thomas Hull (Hull 1997) created a modular origami model F.I.T. that was made of five intersecting tetrahedra and today there are many artists and mathematicians using modular origami techniques to create polyhedra (Gurkewitz and Arnstein 2003). Humiaki Huzita and Jacques Justin discovered seven axioms that are specific to origami. These axioms allow for certain geometric constructions not possible with classical Euclidean Axioms, including trisecting an arbitrary angle (Huzita 1989) and constructing the cubic root of integers (Lang 1996). Robert Lang, a physicist who became one of the foremost origami artists and theorists, has written widely on the mathematical methods for both representational or figurate origami and abstract origami (Lang 2012, 2018).

Folding a piece of paper can be simple and doesn't require any sophisticated tools. However, to model the morphology and to understand the intrinsic properties found in paper folding scientifically is very difficult and requires sophisticated tools of mathematics and computer science. In modelling the origami mathematically, the paper is often assumed to have no thickness (except in the case with thick origami) and doesn't stretch or distort or intersect itself. In reality, paper has thickness and can be bent and distorted in a way that is difficult to describe mathematically. In this chapter, the mathematical methods discussed are aimed at creating origami-inspired functional and aesthetically pleasing lampshades, and so many of the real-world physical models demonstrated might not have exact mathematical descriptions. Physical models can behave in ways that are difficult to understand mathematically, and mathematical models may not have exact analogues in the physical world. For example, in the mathematical study of rigid origami, the mathematical model pretends that the paper is stiff, like sheet metal, and the creases act like hinges. In reality, such rigid origami models can bend in ways the mathematical model won't predict because, in reality, paper can bend.

2. Background on Paper Lanterns

There is a long history of paper lampshade and lantern that have been used both as a functional item for daily life and as a symbolic item for ceremonial occasions. People of many different cultures have long preferred the soft diffused warm light transmitted by translucent materials such as paper and fabric instead of other dazzling light sources (Klint 2018). In western culture, paper or silk lanterns are still used in many traditional festival activities. Traditional luminaria, originating from Hispanic culture and often displayed during Christmas to kindle the spirit of Christ, is made from a paper bag with a folded-down top and filled with a layer of sand that holds a lit candle (Ortega 1973). During the *Festa della Rificolona* in Florence, Italy, decorated paper lanterns carried on sticks become the signature of the festival that is dedicated to the Virgin Mary (Raison 1994). In China, the tradition of making lanterns out of bamboo sticks and paper, silk, skin, and other translucent materials, goes back as early as two thousand years ago. Lanterns or lampshades made out of paper were considered to be especially elegant as they often require high quality craftsmanship. Today, Chinese paper lights can be found as functional objects in homes and as decorative and symbolic items in festival activities such as the lantern festival, a tradition started in the East Han Dynasty when Emperor Ming ordered that lanterns be lit in order to honour the Buddhist spirit during the auspicious full moon period of each new lunar year. Usually Chinese paper lights for homes are designed as a simple spherical or oblong plain form, while the paper lanterns for the festivals are decorated with vibrant colors and elaborate figures from myths in order to enhance the holiday spirit (Song 2015).

The most well-known paper lanterns today are perhaps the Japanese paper lanterns. Lanterns were introduced to Japan from China by Buddhist priests in the 14th century (Hughes 1978). Japanese lanterns, similar to Chinese lanterns, are often made of paper or silk that is stretched over a bamboo stick or wire frame. Different types of paper or silk lanterns are used in different settings for various functions and symbolic meaning. For example, *Chochin*, often in an oblong shape, is used at the entrance of Buddhist temples, in traditional festivals, and at the entrances of bars and restaurants. *Andon*, often in a tetrahedral, cylindrical, or cubic shape, is often used in the interiors of hotels and restaurants. Unlike *Chochin* and *Andon*, *Toro* is only used on special occasions, such as *Toro Nagashi*, the festival of the floating lantern (García 2010). These traditional lanterns were often covered with a thin kozo paper, or occasionally mitzumata paper, that are super strong and yet appear to be light luminous and translucent. The centres of traditional lantern production were often closely link to the product of traditional paper, or called *washi* in Japanese. One of the well-known *washi* making places is the Mino area of Gifu prefecture. The well-known contemporary Akari lights made by Isamu Noguchi, are still produced in Gifu prefecture using traditional lantern-making techniques (Kida 2003).

To create a *Chochin*, a temporary wooden form is used. This form is comprised of several panels that were rotationally and evenly spaced so its silhouette resembles the profile of light. Thin strips of split bamboo are then wound around this wooden form in a spiral. To secure placement of the bamboo strips, string is then looped around each revolution of the bamboo strip vertically. Thin pieces of kozo fiber paper, cut to smaller pieces of rectangles, is applied by brush using thick wheat starch paste and is then trimmed with a knife to follow the curve of the string line. This process is repeated several times till the whole bamboo and wire structure is covered completely with paper. After the paper is dry, the temporary wooden form is then removed as the bamboo strips and the wires remain as the permanent interior structure of the lantern. The paper exterior of the lantern is then adorned with paintings by the artists and finished with lacquered wooden rings on the top and a wood base at the bottom to hold the light source (Nichols, Elgar, and Gausch 2007). Though many great painting masters had created artwork for Chochin in the past, Chochins of the 19th-centuary are often not preserved today as they were often damaged or destroyed during use. One exceptional example was two Chochins that were painted by one of the great ukiyo-e painting master, Katsushika Hokusai (1760-1849), which are preserved at the Museum of Fine Arts, Boston (Nichols, Elgar, and Gausch 2007).

3. Contemporary Origami-inspired Paper Lampshades

As demonstrated in the making of the Chochin lantern, traditional lanterns made from paper or fabric require internal frames to support otherwise flimsy and insubstantial paper and fabric. The process of making the traditional paper lantern is lengthy and tedious and is not suitable for mass production. Today, as the tradition of paper lanterns continues, it is not uncommon to see some contemporary designs use origami-inspired folding techniques to create lanterns or lampshades without internal frames. Folding adds significant structural quality to material. Contemporary origami-inspired light, either folded from paper or from other foldable synthetic material, has become very popular, thanks to a few top-quality lighting designers and manufacturers in the world such as Le Klint and Issey Miyake. Light passes through an origami paper surface, creating a beautiful translucent gradation of light that works well with the modern aesthetic. While the design of origami paper lampshades is largely based on origami design, it also presents new challenges due to its specific set of design constraints as a new genre of functional art.

While traditional paper lanterns often use candle or oil as light sources, other light sources have been developed since the 19th century. Coal-gas lighting was patented in 1804 by German inventor Fredrich Winzer (en.wikipedia.org 2018). Because of the frame ignited by the gas was too intense, functional lampshades made of glass or light fabric were used to attenuate the light. In 1879, Joseph Swan and Thomas Edison invented the first electric light bulb. Again, to disguise the intense electric light, lampshades were used. The lampshades by Louis Comfort Tiffany in coloured stained glass with elaborate patterns for very first electric lights were handmade by skilled craftsmen (Quin and Sibthorp 2012). These Art Nouveau style lamps are still very popular after over a hundred years and the originals have been collected by the museums and the collectors around the world. Since Tiffany's first lampshade, many notable designers, including Frank Lloyd Wright, Josef Hoffman, Gerrit Rietveld, Eileen Gray, Poul Henningson, George Nelson, Alvar Aalto, Isamu Noguchi, just to name a few, have designed lampshades that have become the symbols of iconic contemporary design.

It was unclear when the art of origami was first used in a contemporary lampshade design. In Scala and Sibthorp's Lighting: 20th Century Classics (Quin and Sibthorp 2012), a Le Klint paper shade by the acclaimed and influential Danish architect and industrial designer Kaare Klint that was mass-produced in 1943 was listed as one of the 20th century classics. Kaare Klint's paper folded light can be traced back to his father, P.V. Jesen-Klint, also a well-known architect. As early as 1907, P.V. Jesen-Klint designed a pleated paper lampshade for a paraffin lamp with the help of his friend, captain Jeppe Hagedorn, who had travelled to Japan and learned about the art of origami (Klint 2018). In 1943, Tage Klint, another son of P.V. Jensen-Klint, made modification of P.V. Klint's original shade by adding collars so that the paper shade could fit tightly to a metal stand and founded Le Klint to sell the pleated lampshades commercially. Since then, a series of other leading designers also created pleated paper lampshades for Le Klint, including Peter Hvidt, Orla Molgaard-Nielsen, Erik Hansen, Poul Christiansen, etc. Today Le Klint still produces their lamps by folding one piece of large paper or plastic through handcraft, with the aid of automatic creasing by machines.

Another contemporary origami inspired paper lampshade design example is by Japanese designer Isssey Miyake. Miyake is best known for his origami-inspired fashion designs that can be folded flat and expanded into three-dimensional forms to be worn. Recently Miyake launched a collection of origami-inspired light sculptures called In-Ei, which is made of recycled fiber from PET fibers. Miyake's garment-making techniques are applied in the welding and folding of non-woven plastic fibres to create smooth and seamless joineries (www.isseymiyake.com 2017). In both origami-inspired Le Klint and In-Ei, mathematical principles in both two and three dimensions were explored, resulting in sculptural forms that manipulated the gradations of light and shadows in poetic ways.

4. Light, Origami Design, and Material

When light strikes the mountain and valley creases of a folded surface, it creates dramatic effects of gradations of light and shadows. When a light source is placed behind an origami structure that is folded from translucent material, the light does not pass directly through the material. It diffuses through the material, much like dye diffusing through a liquid. The glaring light source on the other side of the translucent material appears fuzzy and soft when seen from the outside. And it is precisely the perceptual quality of this warm fuzziness that has drawn people of different cultures since ancient times.

A light source positioned in front of an opaque shade will produce various lighting effects such as downlight, uplight, sidelight and backlight. However, when an origami folded design of translucent material is lit from within, the distance of the light source and the material can be essentially neglected; this has minimal effect on the perceptual quality of the light. Areas of the material that receive strong direct illumination tend to dissipate the light by transmitting it to other parts of the object. In order to create more dramatic effects with an origami light, the dihedral angles of the folds need to be carefully considered. The definition of the dihedral angle of an origami fold is the angle measured between two facets. By contrast, the fold angle of an orgami is defined as deviation of the unfolded state. When an origami fold is flat and unfolded, it is deviation from the flatness is 0, and therefore its fold angle is close to 0 and its dihedral angle is close to π. In general, the fold angle and dihedral angle are simply related:

$$\text{Fold angle} = 180° - \text{Dihedral Angle}$$

When the dihedral angles of the folds are large and the fold angle is small, the origami mountains and valleys seem more flat and it is closed to the unfolded state. Decreasing the dihedral angles and increasing the fold angles of the folds to make the folds sharper will bring out more dramatic gradational changes with more contrasts in illumination (Figure 1).

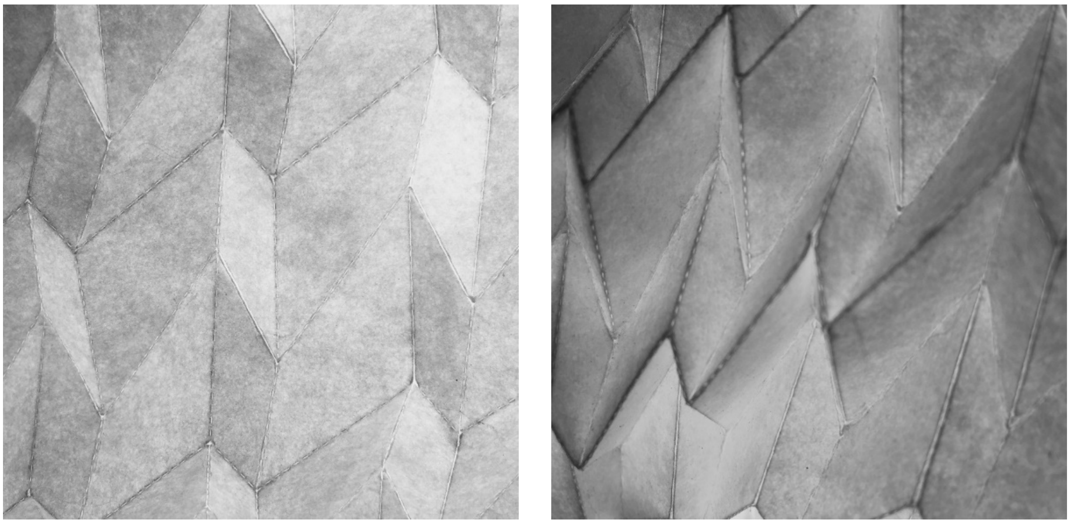


Figure 1. *Comparison of an origami surface folded in paper, lit from behind. (a) Small folding angles of the folds reduce contrast on the illuminated surface. (b) Large folding angles of the folds increase contrast on the illuminated surface.*

Another strategy to increase contrast on a folded origami form illuminated from behind is to vary the material's thickness. Less light will pass through the surface of a material that is double or triple layered, and the illuminated surface will appear to be darker in comparison to areas where there is only a single layer of material (Figure 2). Artist Chris Palmer (Rutzky and Palmer 2011) has used this technique to create his beautiful illuminated *Shadow Folds* by creating origami tessellations on translucent textiles.

The material being considered in this chapter in general is paper or paper-like. Paper is an ideal and versatile choice as it can be easily cut, creased, folded, and rolled, and it can reflect and deflect the light evenly. In particular, all of the folded lampshades photographed in this chapter use a type of hi-tec kozo paper that has a three-layer structure that is perfect for lampshades: a piece of polymer is sandwiched between two pieces of kozo paper. Besides paper, many other materials can be used, including plastic, Mylar, leather, fabric, etc. for the origami lampshade design, as long as they are non-stretchy, paper-like and translucent. However, only paper will be used in this chapter to represent a broader category of material.

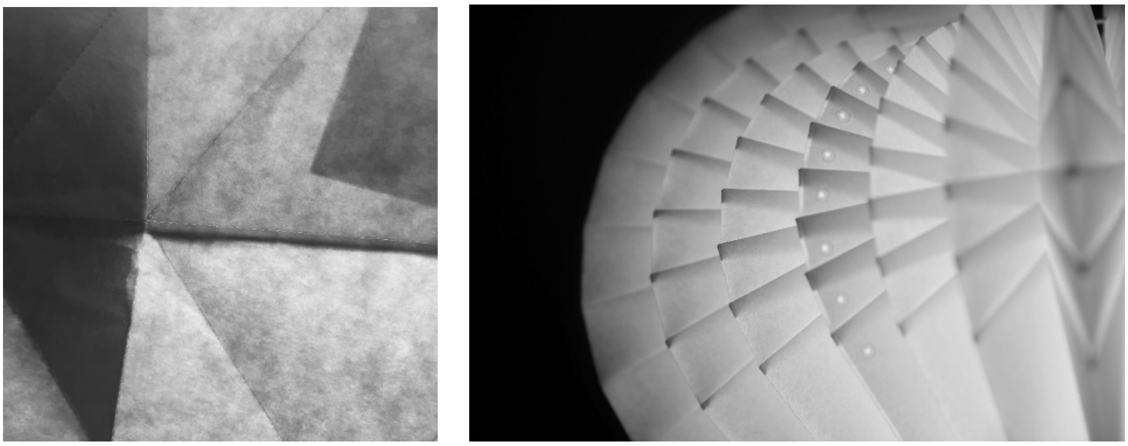


Figure 2. *Examples of using material thickness in illuminated design*. (a) *Illuminated layered paper showing dramatic contrast between areas that are triple-layered, double-layered and single-layered. (b) Folded layered paper can showcase dramatic design features under the illumination.*

5. Design parameters and considerations for origami lampshade design

In this chapter, the origami lampshades discussed must meet the following five design requirements:

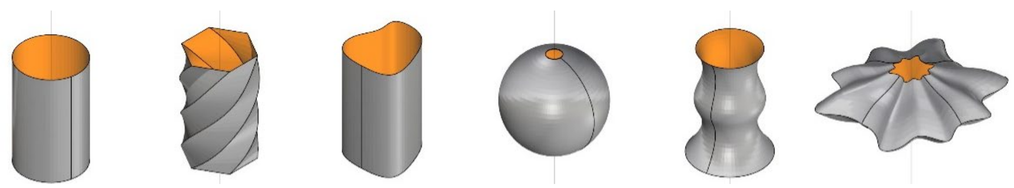
1. The lampshade must be folded from a single sheet of paper. In reality, folding a 1:1 scale lampshade sometimes require a sheet of paper as wide as 10 feet and as long as 30 feet, an impossible feature for sourcing and for digital fabrication. A lampshade is often folded from several sheets of paper that can be connected to form a large flat sheet.
2. The lampshade must be flat-foldable. Flat-foldable design allows compact and sustainable storage and shipping.
3. It must provide a continuously enclosed volumetric body for the light source. The continuously enclosed lampshade should be big enough to allow 4” space between the light source of the lampshade surface, and it should be small

enough allows the light to bounce back and forth between the paper surface to produce a smooth, gentle, and even glow.

4. It must not need to have any additional internal structure support for the 3D volumetric shape other than the origami folds themselves.
5. There must be pleasing contrast in the illuminated origami design.

Since the target lampshades need to be volumetric, several possible shapes are categorized based on Figure 3 below. The first three examples in Figure 3 each has translational symmetry based on a horizontal profile along a vertical axis. Figure 3a is a linear extrusion based on a regular geometric profile, Figure 3b is a helical extrusion based on a regular polygonal profile, and Figure 3c is a linear extrusion based on an irregular geometric profile. The last three examples of Figure 3 each have rotational symmetry based on a vertical profile around a vertical axis. Figure 3d has rotational symmetry based on a symmetric arc, Figure 3e has rotational symmetry based on an irregular geometric profile, and Figure 3f has semi-rotational symmetry as its profile changes as it rotates around a vertical axis. All the lampshade examples discussed in this chapter fall into one of the categories in Figure 3.

Note that some of the polyhedral designs, including the examples in the Platonic solids and Archimedeans solids are left out here. Polyhedral origami lampshades produce aesthetically pleasing geometric designs and involves modular origami techniques, which are outside of the design scope defined here, therefore, they are intentionally left out.



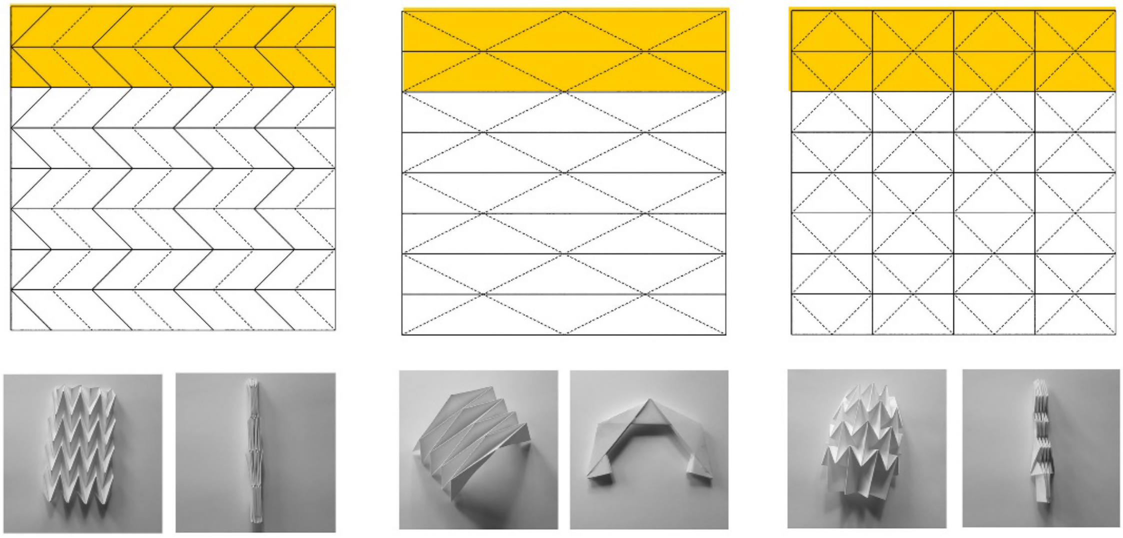
(a) (b) (c) (d) (e) (f)

Figure 3. *Various three-dimensional geometry of lampshade designs: (a) regular extruded cylindrical column, (b) regular helical twist cylindrical column, (c) irregular extruded cylindrical column, (d) regular rotational sphere, (e) irregular rotational surface, (f) irregular rotational surface.*

6. Flat-foldable origami tessellations: Miura, Yoshumura and Waterbomb patterns

Due to the above design parameters and considerations, only flat-foldable three-dimensional tessellations in origami are focused upon. There are many types of origami tessellations including twist tiling, flagstone tessellations, woven tessellation, etc., many of these tessellations and the mathematical methods have been discussed by Lang in his book tilted *Twists, Tilings, and Tessellations* (Lang 2018). In this chapter, only a specific set of tessellations, including the Miura-ori, Yoshimura, and Waterbomb tessellations, that can be folded from a flat sheet and can be flat foldable will be focused. When the fold angles are close to 0°, these three tessellations are unfolded and flat. When the fold angles are close to 180°, these three tessellations are folded and flat. When fold angles are between 0° and 180°, these three tessellations are not completely folded flat and can be formed into a 3D surface. These three tessellations, in many cases, can be used for lampshade designs.

Figure 4 shows these three crease patterns and their folded forms in both their deployable and flat-folded states. In the crease patterns shown in Figure 4 and in the entire chapter, the mountain folds are noted in solid lines and the valley folds are noted in dashed lines. These three tessellations are sometimes called origami corrugations, due to the fact that each of them can be folded into a compact and corrugated form and each of them has corrugated crease lines that are running parallel to each other. The yellow highlights in Figure 4 refer to a single corrugated region of the tessellations, or a strip, that can be modified and generalized. This strip can then be arrayed to form a tessellation that can be either extruded or rotated to form a three-dimensional volume. In each of these tessellations, generalization can only be done in the yellow highlighted region, therefore, this type of generalization is called semi-generalization. Lang studied Miura-ori in detail and laid out a method to semi-generalize the Miura-ori to create various profiles, which he called a semi-generalized Miura-ori (SGMO) (Lang 2018). While the semi-generalizations of these three tessellations provide many design flexibility, some of the interesting designs are the result of combining multiple types of tessellations, such as a tessellation that is a combination of Miura-ori and Yoshimura pattern.



(a) (b) (c)

Figure 4. *Crease patterns and the associated folded forms in deployable states and flat-foldable states: (a) Miura-ori tessellation, (b) Yoshimura tessellation, and (d) Waterbomb pattern.*

7. Mathematical theorems governing flat-foldable origami tessellations

In order to semi-generalize Miura, Yoshimura, and Waterbomb tessellations, the mathematical theorems governing the flat-foldability of these tessellations need to be studied. Various conditions that satisfy the flat-foldability of a crease pattern are discussed below. In general, a crease pattern refers to a set of mountain-folded lines (denoted by solid lines in this article) and valley-folded lines (denoted by dashed lines in this article) appearing when the folded structure is opened flat. A necessary condition for a crease pattern to be flat-foldable locally is given by Kawasaki-Justin Theorem (Mitani 2011, Lang 2018):

Kawasaki-Justin Theorem. *A crease pattern is flat-foldable locally if and only if*

(i) At each vertex (a point where crease lines meet), the number of lines meeting at that vertex is even, and

(ii) the sums of alternating sector angles about that vertex is 180°.

For an entire origami tessellation to be flat-foldable, the Kawasaki conditions must be satisfied by all the inner vertices of a crease pattern. While the Kawasaki's Theorem deals with the number of crease lines and the sector angles at each vertex, it doesn't address the necessary conditions for the mountain or valley assignments for the folds. For a vertex to be flat foldable, both the

Maekawa-Justin Theorem and Big-Little-Big Angle (BLBA) Theorem must also be satisfied.

Maekawa-Justin Theorem. *For any flat-foldable vertex, let M be the number of mountain folds at the vertex and v be the number of valley folds. Then* $M\text{-}V=\pm 2$.

That is, for any vertex, the number of mountain folds and valley folds connected to the vertex must differ by exactly 2.

Big-Little-Big Angle (BLBA) Theorem. *At any vertex, the crease on either side of any sector angle that is smaller than its neighbours must have an opposite crease assignment.*

In other words, at any vertex, the crease assignments for the creases on either sides of any smaller sector angle must be opposite with one being a mountain assignment and one being a valley assignment. And correspondingly, at any vertex, on either sides of the largest sector angle, the crease assignments for creases must be the same with either both being mountain assignments or both being valley assignments.

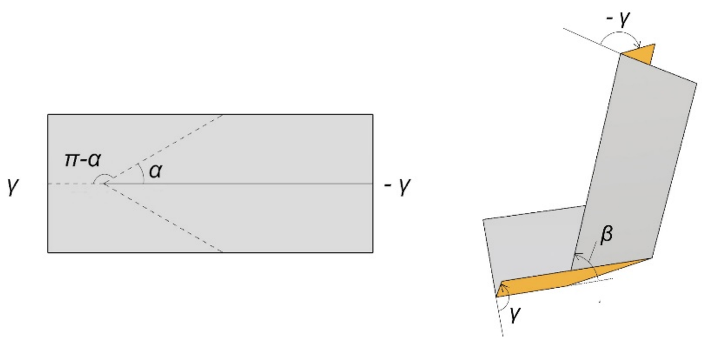
In addition to the sector angles and crease assignments for each interior vertex in a crease pattern, there should be no collision of the parts of the folded structure during assembly. While Lang discussed various scenarios in his book on mathematical methods for geometric origami (Lang 2018), including the Miura-ori, Yoshimura, and Waterbomb tessellations, the geometric strategies in designing crease patterns based on the Miura-ori, Yoshimura, and Waterbomb tessellations, as well as alternations and combinations of these tessellations, for the purpose of specific lampshade designs are discussed in the following sections.

8. Miura-ori Tessellation

The Miura-ori tessellation, credited to Japanese astrophysicist Koryo Miura, has become well-known for its application in deployable structure, such as a solar array in a 1995 mission for JAXA, the Japanese space agency (Miura 2009). The Miura-ori tessellation is made of repeated parallelograms arranged in zigzag formation and has only one type of vertex: a 4-degree ~~of~~ vertex. A key feature of the Miura-ori is its ability to fold and unfold rigidly with a single degree of freedom with no deformation of its parallelogram facets as seen in Figure 4a (Lang 2018).

8.1 Miura-Ori and the bird's-foot vertex

To semi-generalize the Miura-ori so that various shapes of profile paths can be generated, a single strip, or a single corrugated region of the Miura-ori (in the yellow highlighted row in Figure 4a) will be focused upon. ("Semi-generalized Miura-ori" is a term used by Lang to refer to an approach for modifying a Miura-ori to create targeted designs (Lang 2018).) The crease pattern in the semi-generalized strip will then be repeated periodically in another direction to create origami tessellation that can then be folded into either rotationally stretched or vertically extruded the three-dimensional surfaces. At the core of the single strip Miura-ori is the 4-degree vertex, or a bird's-foot vertex that has bi-lateral symmetry (the bi-lateral symmetry allows the semi-generalized Miura tessellation to satisfy the Kawasaki's Theorem). To semi-generalize the Miura-ori strip, three parameters in Figure 5 showing the bird's-foot vertex can be adjusted: the sector angle α, the folding angle along the corrugation crease γ, the bending angle along the corrugation crease β.



(a) (b)

Figure 5. *Crease pattern and three-dimensional geometry of a bird's-foot vertex. (a) crease pattern, (b) partially folded surface.*

The relationship among them as proved by Robert Lang (Lang 2018) can be expressed as:

$$\alpha = \mathrm{ArcTan}\left(\frac{\tan\left(\frac{\beta}{2}\right)}{\sin\left(\frac{\gamma}{2}\right)}\right) \quad . \qquad (1)$$

If γ is constant, decreasing the bending angle β will decrease the sector angle α. If a target profile requires a more shallow bending profile (smaller β) as shown in Figure 4a, it will result in smaller sector angle α. If a target profile requires a sharper bending profile (larger β) as shown in Figure 4c, it will result in a larger sector angle α.

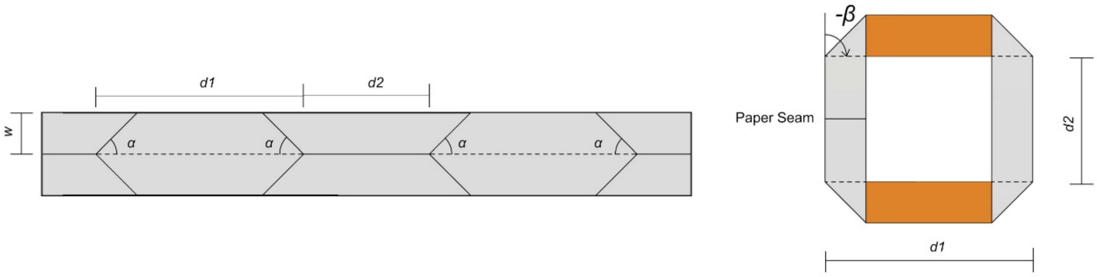
8.2 Folding Miura-ori into cylindrical lampshade with translation symmetry

A Miura-ori strip that can be folded into either a seamless regular polygonal profile or a seamless irregular polygonal profile in which the edges of the folded paper align perfectly. Such a Miura-ori strip can be arrayed into a semi-generalized Miura-ori tessellation and which in turn can be folded into seamless cylindrical lampshade with translation symmetry. To fold the Miura strip into a seamless profile outlined by regular polygons with n sides (Figure 6) when $\gamma = 180^{\circ}$, angle α and β must satisfy condition (1) and (2) below respectively:

$$\alpha = \frac{\pi}{n} \qquad (2)$$

$$\beta = 2\alpha = \frac{2\pi}{n} \qquad (3)$$

It is important to note that n must be an even number in this specific generalization, and therefore, the regular polygons must have an even number of sides. If the polygons have odd number of sides, the mountain or valley crease assignments on the corrugation creases at the both ends of a Miura strip won't be able to match.



(a) (b)

Figure 6. *A Miura strip that folds into a square profile with $\alpha = 45^{\circ}$ and $\beta = 90^{\circ}$. (a) crease pattern, (b) folded geometry with $\gamma = 180^{\circ}$.*

In the above example, let w to be half of the width of the strip, let d_1 and d_2 be the distances between the three consecutive vertex points on the Miura strip. To fold the strip into a seamless profile in the shape of a square when $\gamma = 180^{\circ}$, the distances between next three consecutive vertex points also need to be d_1 and d_2, and in addition, the relationship between d_1 and d_2 must satisfy the condition below as the side lengths of the square profile must be equal:

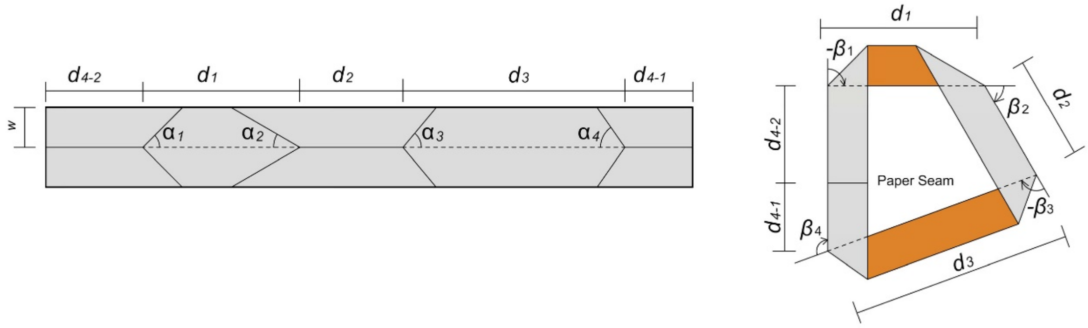
$$d_1 = d_2 + 2w/\mathrm{Tan}(\alpha) \qquad (4)$$

To fold the Miura strip into a seamless profile outlined by irregular polygons with n sides (Figure 7) when $\gamma = 180°$, let the number of a bird's-foot vertex in a Miura strip be i. Each sector angle α_i (as in α_1, α_2, α_3 and α_4 in Figure 7a) and each bending angle β_i

(as in β_1, β_2, β_3 and β_4 in Figure 7a)must satisfy conditions (3) and (4) below respectively:

$$\sum_{i=1}^{n} \alpha_i = \pi \qquad (5)$$

$$\sum_{i=1}^{n} \beta_i = (n-2) * \pi \qquad (6)$$



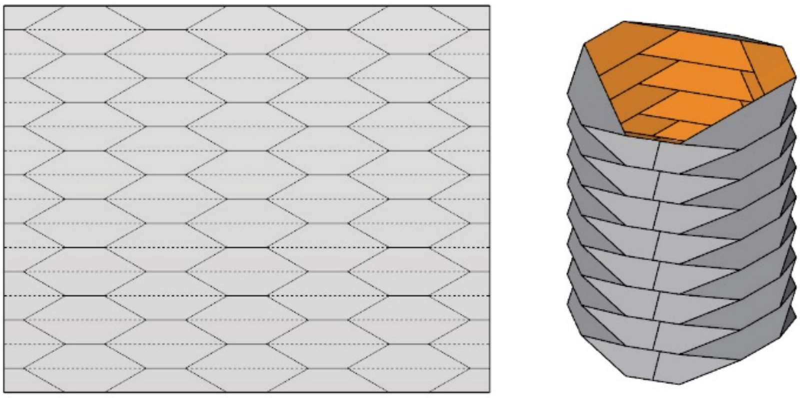
(a) (b)

Figure 7. *A Miura strip that folds into an irregular quad profile when completed folded flat with α_1= 45°, α_2= 30°,α_3= 50°, and α_4= 55° : (a) crease patter, (b) folded geometry.*

To fold the above Miura strip into a flat seamless design with a profile of an irregular quad, the distances between the vertex points are all different, however, two of the distances cannot be arbitrary. In the example above, if d_1 and d_2 pre-determined, d_3 and d_4 must be derived from a value based on d_1, d_2, w and α_i.

The Miura strips in both Figure 6 and 7 can be arrayed in vertical direction to create the Miura tessellation. In general, these tessellations can be folded into seamless cylindrical columns physically in paper that are suitable for lampshade application. As γ increases, the folded surface will bring out more dramatic gradations changes with more contrasts in illumination. However, the paper lampshade designed using the above calculations are not mathematically accurate. From equation 2 and 3, for a Miura strip to be folded into a profile in the shape of regular hexagon when γ=180°, α needs to 30° and β needs to be 60°. However, from equation 1, for γ=150°and α=30°, β is about 53°, which is less than 60° that is needed to for the regular polygonal profile to be a hexagon. Instead, when γ=150° and β=60°(a necessary condition for folding a seamless regular hexagonal column), α needs to be 33.69° based on equation 1.

Figure 8 shows a mathematically correct origami model that is folded into a cylindrical column based on a regular hexagonal profile. Note that the sector angle α =33.69° while β=60° and γ=150°. For this mathematical model to stay valid, γ needs to be 150°, which means that the structure cannot be deployed or collapsed. Whereas in a paper model of similar shape, γ can be changed because of the bending and distortion in the paper, thus allowing the structure to be collapsed and deployed in a way that is mathematically impossible.



(a) (b)

Figure 8. *A mathematically correct Miur-ori hexagon column and its crease pattern. (a)Crease pattern with sector angle α =33.69°. (b) Folded cylindrical lampshade with β=60°, γ=150°.*

8.3 Folding Miura-ori into a lampshade with rotational symmetry

While a Miura-ori strip can be used generate various seamless and continues profile, a Miura-ori strip can also be used generate expressive open profile. This Miura-ori strip can then be arrayed to create a Miura-ori tessellation which can then be folded and stretched to create three-dimensional surfaces with rotational symmetry. The original Le Klint origami lamp, modified by Tage Klint (Figure 9), was folded with this method.

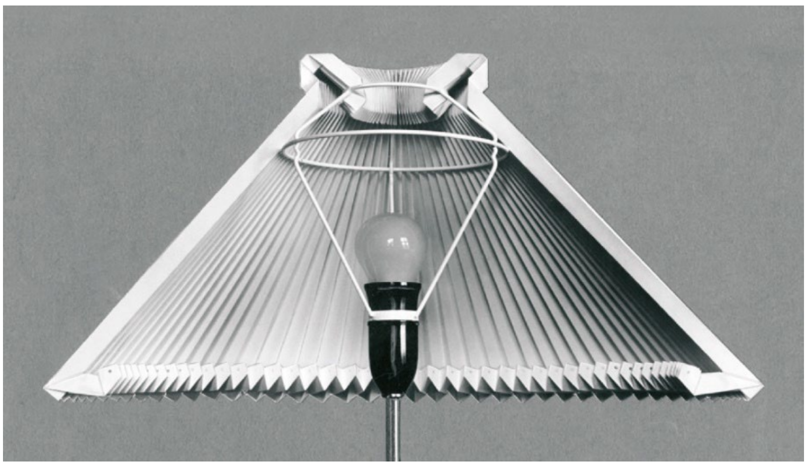
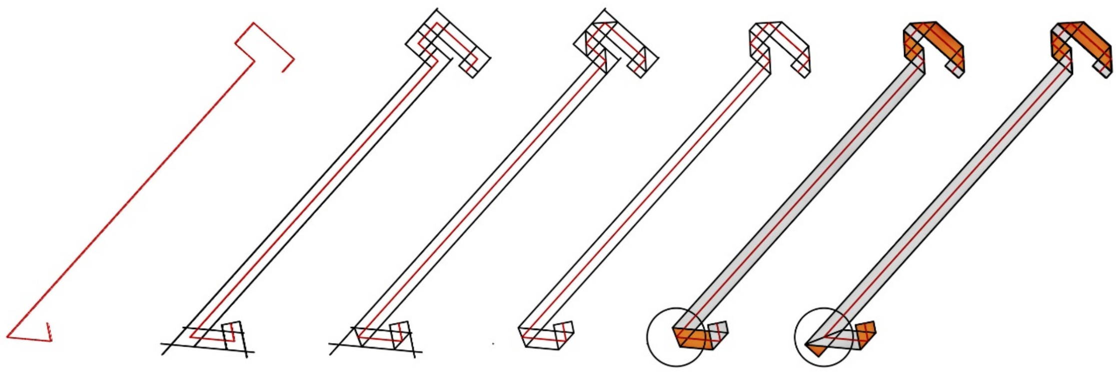


Figure 9. *An early mass-produced Le Klint light design by Tage Klint. Photo Courtesy of Le Klint.*

In Lang's book (Lang 2018), Lang detailed a construction process for using Miura-ori to generate any target profile that can be used to create a three-dimensional surfaces, which he called a semi-generalized Miura-ori (SGMO). Lang's construction method is fairly straightforward and requires almost no mathematics. Below are the steps of the construction process of Tage Klink's lamp using a method that is similar to Lang's method with a small modification using divots at bottom rim of the Klint lampshade (Figure 10).

1. Draw a Le Klint lampshade profile (Figure 10a).
2. Draw pairs of lines parallel to and equally spaced from the original profile. Make sure the lines are long enough to intersect with the next pair of lines. The offset distance between the original profile and its parallel lines is half of the pleating width *w* (Figure 10b).
3. Draw diagonal lines connecting the turning points on the profile and the respective intersecting points (Figure 10c).
4. Trim the access guidelines (Figure 10d).
5. Draw the silhouette of the folded strip follows the desire path (Figure 10e).
6. Change the bird's foot vertex to a bird's foot vertex with the divot (Figure 10f).



(a) (b) (c) (d) (e) (f)

Figure 10. *Graphical construction of a Miura-ori strip to be folded into the profile of the Le Klint lampshade design by Tage Klint.*

The folded strip in Figure 10f above can be rearranged onto a flat sheet in the form of polygons (Figure11a). This pattern can then be mirrored. Mountain and valley creases can then be assigned as in Figure 11b. The pattern in Figure 12b can then be arrayed to generate the crease pattern in Figure 11c that can be folded into the Le Klint lamp with rotational symmetry designed by Tage Klint.

__Figure 11.__ Graphical construction of the crease pattern that can be folded into the Le Klint lampshade design by Tage Klint: (a) unfolded strip, (b) mirror patterns, (c) arrayed crease pattern.

9. Yoshimura Tessellation

The Yoshimura tessellation was discovered by scientist Y. Yoshimura (Yoshimura 1955) while he was researching the buckling patterns of thin-walled cylinders. One of the most important features of the Yoshimura pattern is its ability to allow the form to reduce the dimensions in all directions when compressed or folded, facilitating easy transportation and storage. A regular deployment of the pattern produces an approximated arc form that is suitable for creating three-dimensional volumetric surface.

9.1 Yoshimura tessellation and its double bird’s foot vertex

If two Miura-ori bird’s foot vertices that have opposite sector angles come close together to form a single vertex, it becomes a double bird’s foot vertex. This double bird’s foot vertex can be arrayed on a strip and the strip that can then be arrayed to form a Yoshimura tessellation (Figure 4b). Since the double bird’s-foot vertex is a degree-6 vertex which has three degree of freedom, the deployment and compression of Yoshimura is somewhat flexible. If the boundary of the Yoshimura pattern is not fixed, then its folded surface can be twisted and deformed in multiple directions (Wu 2017).

To semi-generalize the Yoshimura pattern so that various shapes of profiles for either extrusion or rotation can be generated to be used as a lampshade, a single strip of Yoshimura pattern will be focused upon. At the core of the single strip Yoshimura is the double bird’s-foot vertex that has bi-lateral symmetry (the bi-lateral symmetry allows the semi-generalized Yushimura tessellation to satisfy the Kawasaki's Theorem). Since a double bird's-foot vertex is a degree-six vertex that has three-degree of freedom, folding angle γ on the left and right could work independently. However, to restrict the degree of freedom in the mechanical behaviour of a Yoshimura strip, only the consistent folding angle γ is considered here. To semi-generalize the Yoshimura, four parameters in Figure 12 showing the double bird's-foot vertex can be adjusted: the two sector angles α_1 and α_2, the folding angle along the corrugation γ, the bending angle of along the corrugation β.

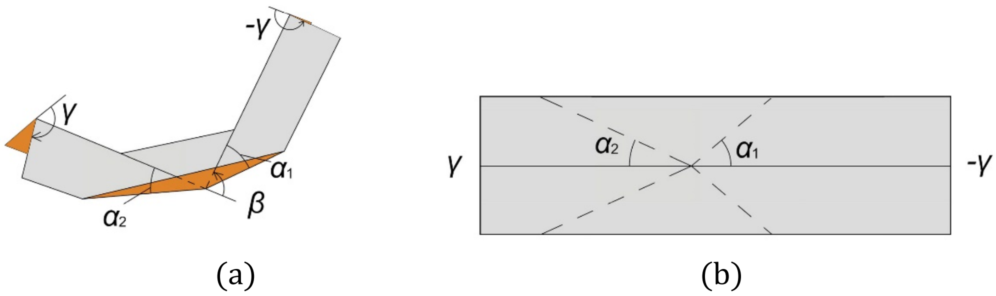


Figure 12. *A Yoshimura double bird's-foot vertex: (a) crease pattern, (b) partially folded form when* $\alpha_1 = 37.7^{\circ}$, $\alpha_2 = 21.7^{\circ}$, $\beta = 95.0^{\circ}$, and $\gamma = 99.9^{\circ}$.

From equation 1 of a bird's-foot vertex, equation 7 can be arrived as proved by Robert Lang (Lang 2018):

$$\tan\left(\frac{\beta}{2}\right) = \tan(\alpha)\sin\left(\frac{\gamma}{2}\right) \qquad (7)$$

In a double bird's foot vertex, the folding angle γ is restricted to be consistent. From equation 7, β in the double bird's foot vertex must therefore satisfy the equation 8 below when two sector angle α_1 and α_2 are different:

$$\beta = 2Arctan(Tan(\alpha_1) \times Sin\left(\frac{\gamma}{2}\right)) + 2Arctan(Tan(\alpha_2) \times Sin\left(\frac{\gamma}{2}\right)) \qquad (8)$$

9.2 Folding Yoshimura into cylindrical lampshade with translational symmetry

A Yoshimura strip that can be folded into either seamless a regular polygonal profile or a seamless irregular polygonal profile. Such a Yoshimua strip can be arrayed into a Yoshimura tessellation and which in turn can be folded into seamless cylindrical lampshade with translation symmetry. The seamless condition results from the fact that the edges of the folded paper align perfectly without any gaps. To fold the Yoshimura strip into a seamless profile outlined by regular polygons with n sides (Figure 13) when $\gamma = 180^{\circ}$, angle α and β must satisfy equations (9) and (10) below respectively:

$$\alpha = \frac{\pi}{n} \qquad (9)$$

$$\beta = 4\alpha = \frac{4\pi}{n} \qquad (10)$$

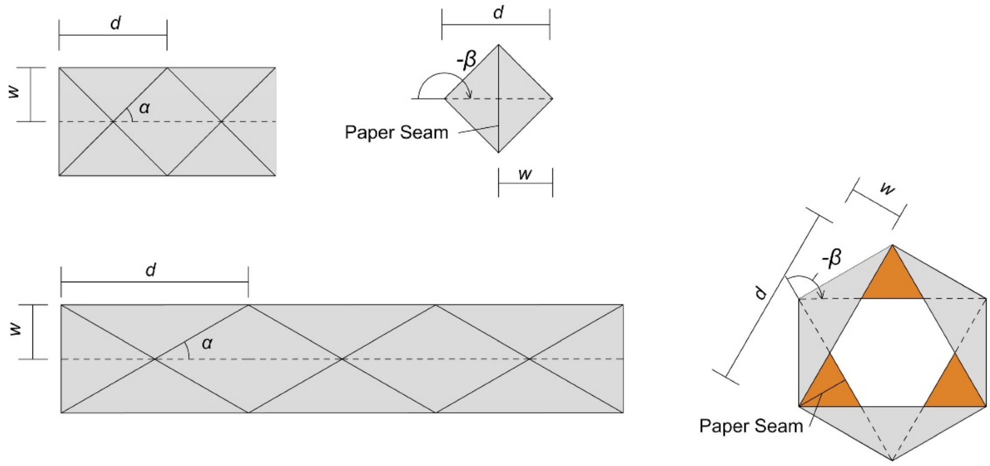


Figure 13. *Two Yoshimura strips: (a) One folds into a square profile with α = 45°, β = 180°, and γ=180°, (b) the other one folds into a hexagon profile with α = 30°, β = 120°, and γ=180°.*

It is important to note that n must be an even number that is bigger than 4. If the polygons have an odd number of sides, the mountain or valley crease assignments at the ends of a Yoshimura strip won't be able to match. When n=4 and $\alpha = 45°$ the Yoshimura strip can be folded with no overlapping part as in Figure 20a. However, the folded strip doesn't leave any space for a light source if the strip is arrayed and folded into a lampshade. When n=6, $\alpha = 30°$, $w = \frac{\sqrt{3}}{6}d$, the ratio between the area of the empty space and the over enclosed area by the folds is 1/3. The lampshade folded from the tessellation that is arrayed from such a strip is big enough to host a light source. In general, in a Yoshimura regular polygonal strip, let A_L to be the larger enclosed area by the folds, and A_S to be the empty space. Since

$$A_L = \frac{d}{2\mathrm{Cos}(\alpha)} \qquad (i)$$

$$A_S = \frac{d * \mathrm{Tan}\,(\alpha)}{\mathrm{Tan}(2\alpha)} \qquad (ii)$$

Based on the above two equations, the ration between A_S and A_L is therefore given by the following equation:

$$\frac{A_S}{A_L} = f(\alpha) = \frac{2Sin(\alpha)}{Tan(2\alpha)} \qquad (11)$$

Plotting the function $f(\alpha)$ gives us the following diagram (Figure 13). As the sector angle α gets smaller, the ration between A_S and A_L gets larger, which means that the empty volume enclosed by the paper folds gets bigger compared to the overall paper fold volume. The empty volume that is a big enough to host a light source is a desired feature for a lampshade. However, as α gets smaller and the empty volume gets larger, the pleating width *w* is also getting smaller. Smaller pleating width creates less dramatic gradation of light and shadows. Therefore, the key here is to balance these design considerations to have a successful lampshade design.

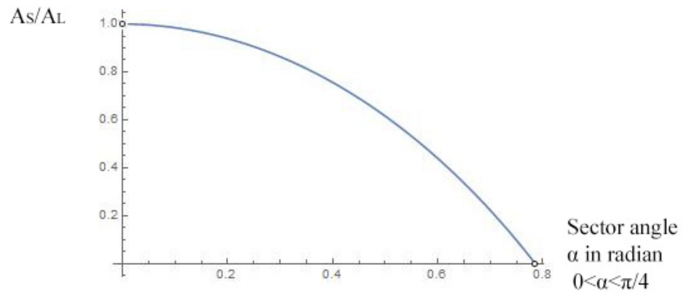


Figure 14. *Graph show the ration between A_S and A_L and Sector angle α.*

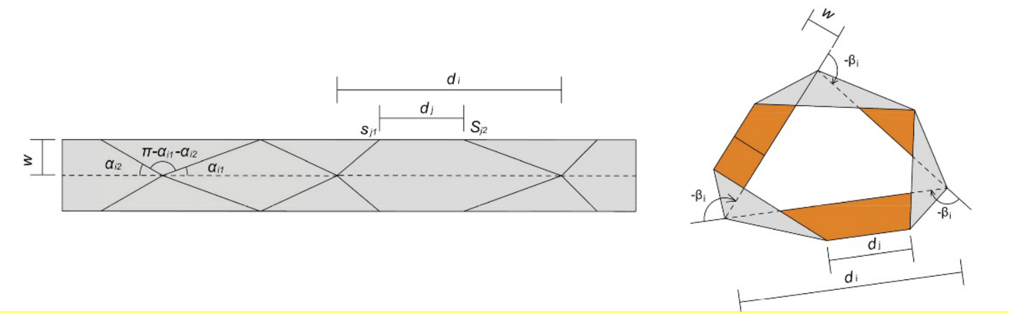
A Yoshimura strip can be folded into a seamless profile outlined by irregular polygons with *n* sides (Figure 15) when γ =180°. Let the number of double bird's-foot vertex in a Yoshimura strip be *i*. Let the distance between the two consecutive double bird's-foot vertex be d_i. If the lines that divide sector angles of two consecutive double bird's-foot vertex don't meet together, then there are two points (for example, points S_{j1} and S_{j2} in Figure 22) at pleating edge. Let the number of such instances in the strip be *j*. The distance between the two points to be d_j. Therefore, $n= 2i + j$. The angle α_{i1}, angle α_{i2}, and β_i must satisfy condition (12) and (13) below respectively:

$$\sum_{i=1}^{\frac{n-j}{2}} \alpha_{i1} + \alpha_{i2} = \pi \qquad (12)$$

$$\sum_{i=1}^{\frac{n-j}{2}} \beta_i = 2\pi \qquad (13)$$

The relationship between d_i and d_j can be expressed below:

$$d_i = d_j + \frac{w}{Tan(\alpha_{i1})} + \frac{w}{Tan(\alpha_{(i+1)2})} \qquad (14)$$



(a) (b)

Figure 15. *A Yoshimura strip folds into an irregular octagon with sector angles α to be 30°, 20°, 25°, 40°, 20° and 45° respectively: (a) crease pattern, (b) folded profile.*

In addition, for the Yoshimura strip to be folded into a seamless n-gon, not all of the distances d_j can be arbitrary. (In general, to construct an n-gon, all of the interior angles and n-2 consecutive side lengths need to be determined, and the remaining two sides can be calculated based on the interior angles and n-2 consecutive side lengths.) In a Yoshimura strip, when $d_j = 0$, d_i is determined by w and sector angles α_{i1} and α_{i2}. Therefore, for the Yoshimura strip to be folded into a seamless n-gon, the length of at least two consecutive d_j can't be arbitrary.

The Yoshimua strip can also be arrayed to create the Yoshimura tessellation. In general, these tessellations can be folded into seamless cylindrical columns physically in paper that are suitable for lampshade application. However, similar to the examples with a Miura tessellation, the paper lampshade designed using the above calculations are not mathematically accurate. The distortions in paper allow the cylindrical structure to be collapsed and deployed in a manner that is mathematically impossible.

9.3 Folding Yoshimura into a lampshade with rotational symmetry

Similar to a Miura-ori strip, a Yoshimura strip can also be used to generate an expressive open profile. This strip can then be arrayed to create a Yoshimura tessellation that can be folded and stretched to create three-dimensional surfaces with rotational symmetry. The first mass-produced Le Klint origami lamp, designed by Kaare Klint (Figure 16), was folded using a Yoshimura pattern.

Figure 16. *The first mass-produced Le Klint lamp designed by architect Kaare Klint. Photo Courtesy of Le Klint.*

To generate a Yoshimura strip based on any arbitrary profile curve, almost no mathematics is involved. Below is a construction process of generating a Yoshimura folding pattern to be used to fold a Yoshimura strip based on the Le Klint origami lamp above.

1. Draw a desired Le Klint lamp profile in a curve BC that is connected to a straight line AB (Figure 17a).
2. Choose an arbitrary point D on curve BC. Choose an arbitrary pleating width w. The larger the *w*, the less smooth will be the final lampshade profile design. Draw a circle using D as the center point and *w* as the radius. Draw line BE and let BE be tangent to the circle and let E be on the curve BC. Draw lines DB and DE (Figure 17b).
3. Draw another circle using E as the center and *w* as the radius. Repeat step 2 till all the lines are drawn (Figure 17c).
4. Offset line AB at the distance of *w* to the left to create line FG´. Line EB intersects with line FG´ at the point G (Figure 17d).
5. Trim the access line GG´ and draw the silhouette of the folded strip so that it follows the desired path (Figure 17e).

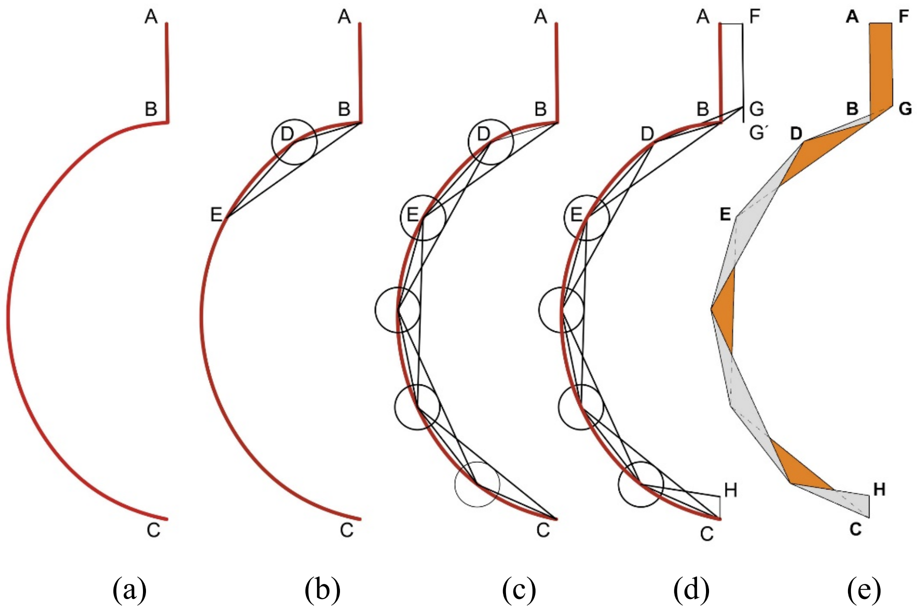


Figure 17. *The process of construct Yoshimura strip based on the lamp designed by Kaare Klint.*

The Yoshimura strip above then can be arrayed and stretched to form a three-dimensional Lampshade surface (Figure 18).

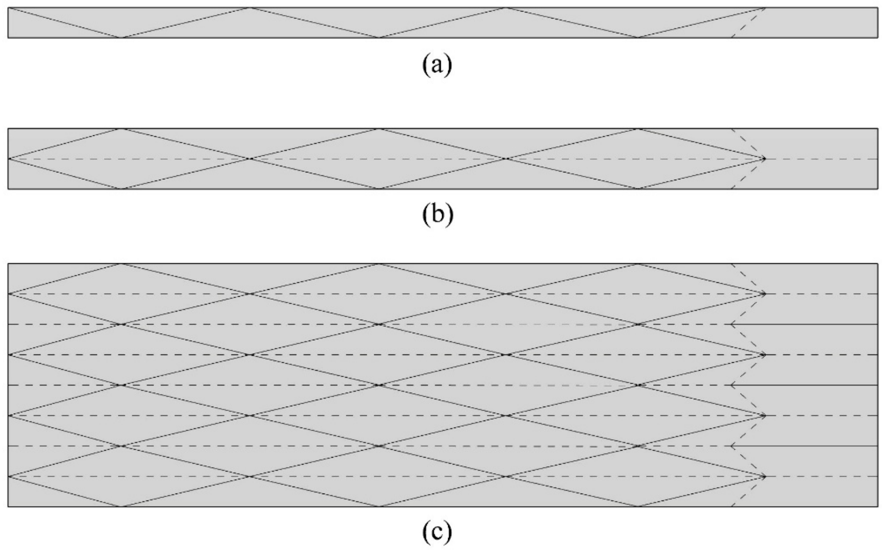


Figure 18. *Graphical construction of the crease pattern that can be folded into the Le Klint lampshade design by Kaare Klint: (a) unfolded strip, (b) mirror patterns, (c) arrayed crease pattern.*

Similar to the rotational stretch Miura-ori or Miura-ori with divots added, the rotationally stretched Yoshimura in the paper model is a distortion when compared with a mathematically correct model. With its rotationally stretched pleats, the bending angle β varied based on the changes in folding angle λ, the radial distance from the axis of rotation, the number of arrays used in the rotational form, and the distortion and bending of the paper.

10. Waterbomb Tessellation

In recent years, Waterbomb tessellation has become very popular due to the well-known “magic ball,” designed by Yuri Shumokov (Shumakov and Shumakav 2015). Yuri and Katrin Shumokov have developed a wide variety of models, from dragon’s egg and hot air balloons to lampshades, that are based on the Waterbomb tessellation. However, the waterbomb tessellation can be traced back to the great Japanese master of geometric folding Shuzo Fujimoto about four decades ago. In his masterwork *Rittai Origami*, Fujimoto described the Waterbomb tessellation (Fujimoto 1976). Perhaps the most well-known application of the Waterbomb tessellation was made by Zhong You and Kaori Kuribayashi et. al. (Kuribayashi et al. 2006) when they applied the tessellation in the design of a heart stent.

10.1 Waterbomb tessellation and its vertices

The Waterbomb tessellation (see Figure 4c) is known for its arrays of square grids, each of which is composed of a traditional Waterbomb base. A traditional Waterbomb base is essentially a double bird’s foot vertex that is similar to the vertex in a Yoshumura pattern. In a Waterbomb tessellation, each row of the Waterbomb base is offset by half of a Waterbomb unit from the rows adjacent to it. Many researchers have studied Waterbomb tessellation and have learned that it displays very interesting mechanical behaviours (Lang 2018). Since a Waterbomb tessellation has all of its vertices as degree-6 vertices, an overall tessellation has many degrees of freedom. However, a physical model folded from a Waterbomb tessellation has a preferred state that is the result of an equilibrium found by balancing the springiness of the folds found in the material and the movement constrained by the folding crease pattern. The springing and morphing surface of a magic ball designed by Yuri Shumokov is a good example of the Waterbomb tessellation’s interesting mechanical behaviour.

Similar to the Miura tessellation and the Yoshimura Tessellation, to semi-generalize the Waterbomb pattern so that various shapes of profiles for either extrusion or rotation can be generated, a single strip of Waterbomb pattern (yellow highlighted region in Figure 4c) will be focused upon. At the core of the single strip Waterbomb tessellation is a basic Waterbomb tessellation unit that has a six-degree double bird’s-foot vertex and two six-degree flat bird’s foot vertexes, each of which has bi-lateral symmetry (the bi-lateral symmetry allows the semi-generalized Waterbomb tessellation to satisfy the Kawasaki's Theorem). To semi-generalize the Waterbomb tessellation, several parameters in Figure 19 can be set up: the four sector angles α_1, α_2, α_3, and α_4, bending angles β_1, β_2, and β_3, and folding angle γ. (There are four folding angles of the corrugation creases OA, AE, OB, and BD. Since a Waterbomb strip has many degrees of freedom, to simplify the mechanical behaviour of a Waterbomb strip, the degree of freedom in a Waterbomb is reduced by restricting the four folding angles of the corrugation crease OA, AE, OB, and BD to have the same value of γ.) Bending angles β_1, β_2, and β_3 are the bending angles of the corrugation creases. β_1 is the bending angle between creases BO and OA, β_2 is the bending angle between creases OA and AE, and β_3 is the bending angle between creases OB and BD. To fold a basic unit of the Waterbomb tessellation, the relationship between folding angle γ, sector angle α_1, α_2, α_3, and α_4, bending angle β_1, β_2 and β_3 must all satisfy the certain conditions. Note that β_1 corresponds with a double bird's-foot vertex, therefore β_1 shares the same condition as equation 8.

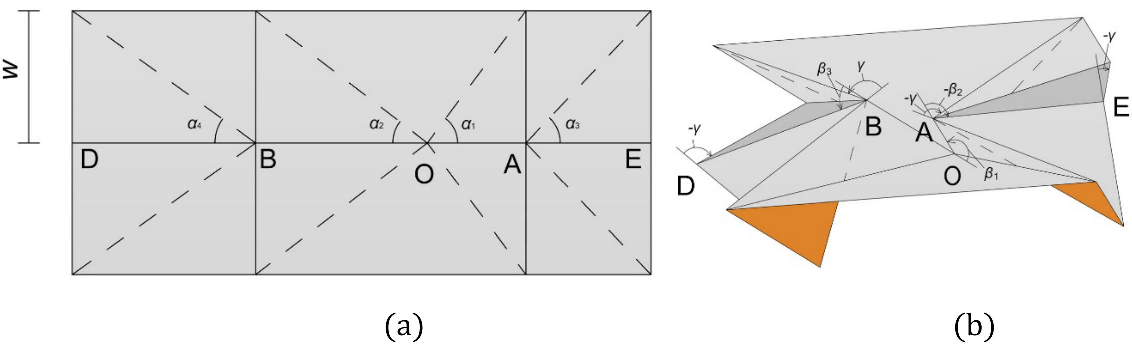


Figure 19. *A Waterbomb basic unit: (a) crease pattern (b) partially folded when* $\alpha_1 = 53.2°$, $\alpha_2 = 37.5°$, $\alpha_3 = 46.5°$, $\alpha_4 = 35.7°$, $\beta_1 = 167.3°$, $\beta_2=94.3°$, $\beta_3=115.4°$, and $\gamma=123.4°$.

If the folding angles of each of the corrugation crease are allowed to change, the Waterbomb strip can be bent into various curves. This characteristic allows a Waterbomb tessellation, arrayed from a Waterbomb strip, to bend in both the direction of along the corrugation folding direction and the direction perpendicular to it.

10.2 Folding Waterbomb Tessellation into cylindrical lampshade with translational symmetry

A Waterbomb strip that can be folded into either a seamless regular polygonal profile or a seamless irregular polygonal profile. Such a Waterbomb strip can be arrayed into a Waterbomb tessellation, which in turn can be folded into a seamless cylindrical lampshade with translation symmetry. To fold a Waterbomb strip into a seamless profile outlined by a regular polygon with *n* sides (the polygons here are more like star-shaped polygons) when γ =180°, angle α and β must satisfy conditions (15) and (16) below respectively:

$$\alpha = \frac{\pi}{4} + \frac{\pi}{2n} \qquad (15)$$

$$\beta = \pi - 2\alpha \qquad (16)$$

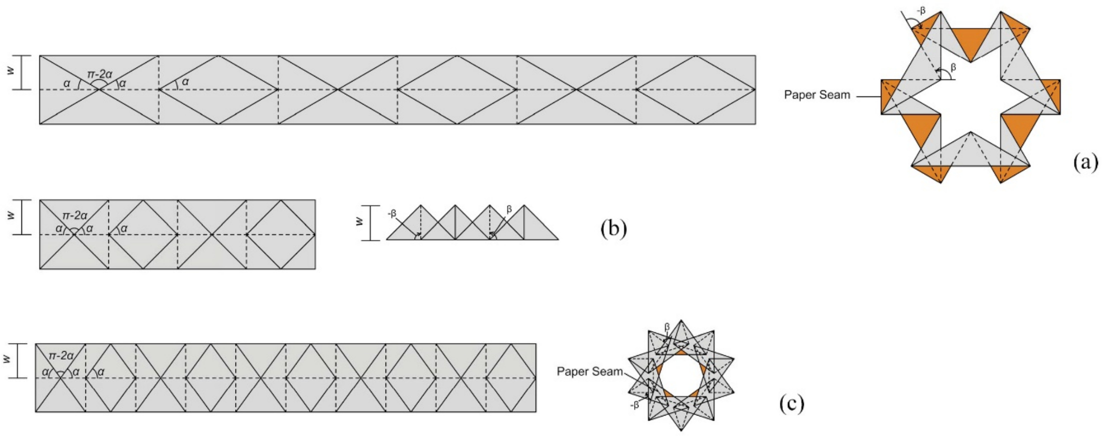


Figure 20. *Three Waterbomb strips: (a) folds into a star hexagon profile with α = 30°, β = 120°, and γ=180°, (b) folds into a linear profile with α = 45°, β = 90°, and γ=180°, (c) folds into a linear profile with α = 54°, β = 72°, and γ=180°.*

Note that for conditions above, n must be even number, and α must be <= 45° . Figure 20 above shows three Waterbomb strips. When α = 30°, the strips folds into a seamless star-shaped hexagon. When α = 45°, however, the Waterbomb strip can't be folded into a seamless star-shaped hexagon, but instead it is folded into a linear strip. When α = 54°, the Waterbomb strip folds into a seamless star-shaped decagon. However, the Waterbomb tessellation derived from the array of this Waterbomb strip doesn't result in an enclosed area that is large enough to host a light bulb, therefore it is not suitable for lampshade application.

Figure 21 shows a lampshade that is folded from a waterbomb tessellation that is made from an array and translation of this Waterbomb strip from Figure 19a. A Waterbomb tessellation has many degrees of freedom. Unlike a Miura-Ori or a Yoshimura tessellation in which the degree of freedom is zero when the boundaries of the tessellation are closed, the degree of freedom in a waterbomb tessellation with its edges connected still has more than one degree of freedom, thus allowing the deployed form to flex into various curvilineal forms in the direction of translation.

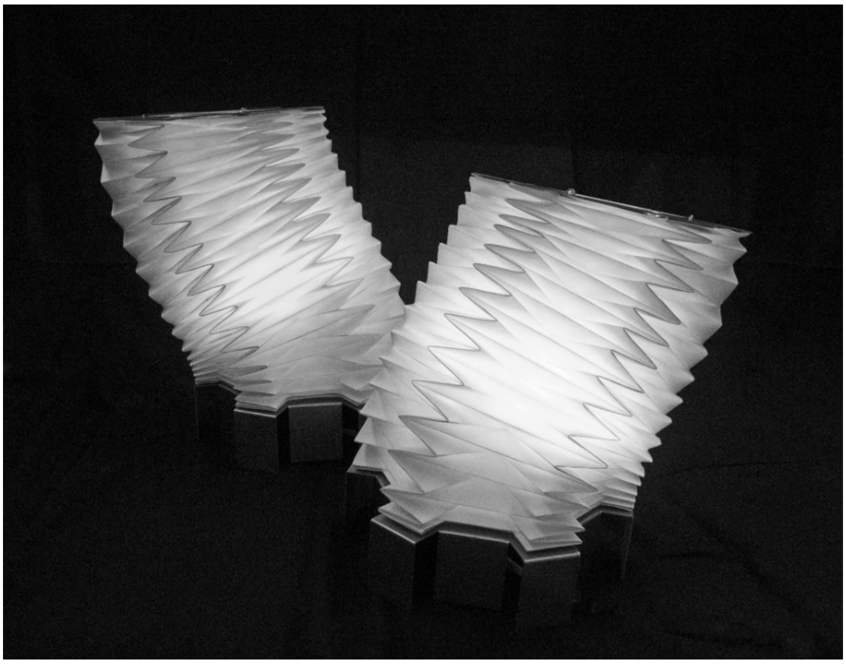


Figure 21. *An example lampshade with translational symmetry based on Waterbomb tessellation. Note that the degree of freedom in a Waterbomb tessellation allows the lampshade to bend in various curvilinear forms.*

10.3 Folding Waterbomb tessellation into a lampshade with rotational symmetry

Similar to a Miura-ori strip and a Yoshimura strip, a Waterbomb strip can also be used to generate an expressive open profile. This strip can then be arrayed to create a Waterbomb tessellation that can be folded and stretched to create three-dimensional surfaces with rotational symmetry. When a Waterbomb pattern is bent into a curve, it curves transversely and forms a synclastic shape that is similar to a spherical surface with rotationally symmetry (Lang 2018). Origami artists Yuri and Katrin Shumokov have used this method to create many lampshades with rotational symmetry. However, a Waterbomb tessellation can also be anticlastic, in other words, if it is bent in one direction then it will curve transversely in the opposite direction, similar to the surface of a saddle. The difference here is due to the variation in the sector angle α.

Figure 22 shows two Waterbomb tessellations. Each of the Waterbomb tessellations shares a consistent sector angle α. In the first Waterbomb tessellation, the sector angle α=45; in the second Waterbomb tessellation, the sector angle α=52°. The two Waterbomb tessellation surfaces are formed when the corrugation fold angle γ is 75°, 104° and 131°. Notice in the first Waterbomb tessellation that the surfaces are mostly synclastic, while in the second Waterbomb tessellation, the surface become anticlatistic when γ=131°. In general, when α> 45°, the Waterbomb tessellation displays an anticlastic feature. Since a Waterbomb tessellation has many degrees of freedom, a physical model folded from the Waterbomb tessellation is often the result of an equilibrium found by balancing the springiness of the folds and the constraint set upon by the folding pattern and the boundary condition. This characteristic, when combined with both the synclastic and anticlastic natures of a waterbomb tessellation, allows a waterbomb tessellation to be shaped into a surface that has both a positive curvature and a negative curvature, similar to a torus.

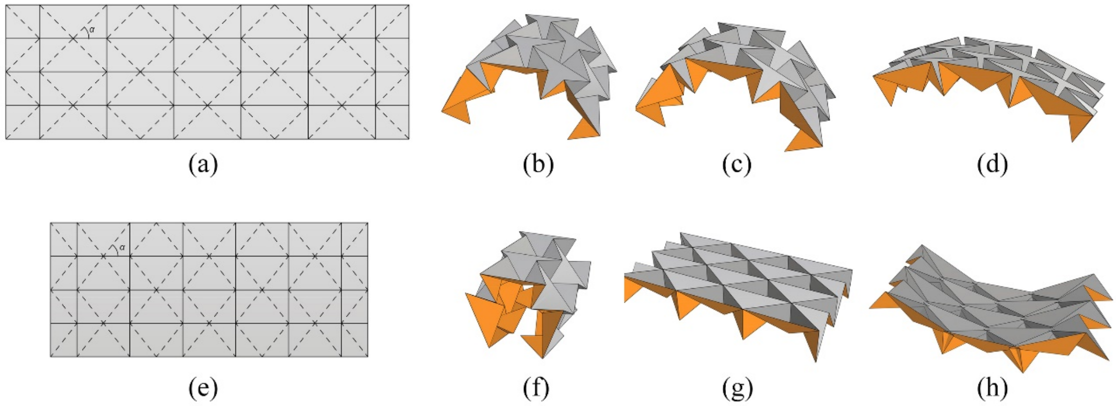
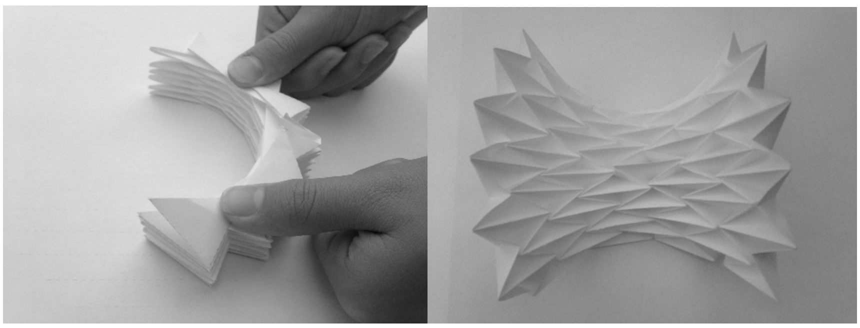


Figure 22. *Comparison of two Waterbomb tessellations: (a) is the crease pattern of (b), (c), (d) with sector angle α=45°, (e) is the crease pattern of (f), (g), (h) with sector angle α=52°. In (b) and (f), the corrugation fold angle λ= 75°; in (c) and (g), the corrugation fold angle λ= 104°, and in (d) and (h), the corrugation fold angle λ= 131°.*

Figure 22 shows a torus-shaped lampshade folded from a Waterbomb tessellation with all of the sector angle α consistent and larger than 45°. When γ =180°, the Waterbomb tessellation folded into a semi-circular open profile (Figure 23a). As γ decreases, the Waterbomb tessellation deploys to a surface that is doubly-curved (Figure 23b), which allows all its boundaries to close and form into a Torus as shown in the picture (Figure 23c).

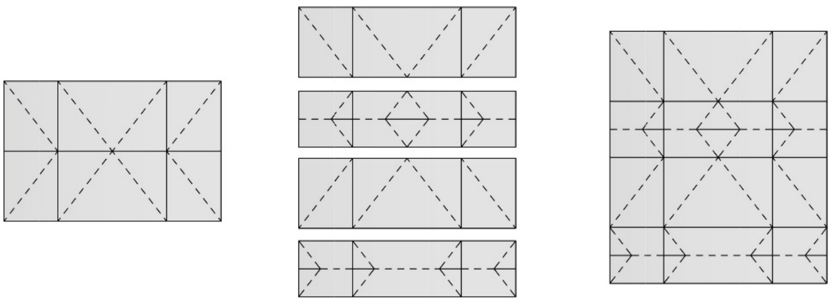


(a) (b)

(c)

Figure 23. *A lampshade with rotational symmetry based on Waterbomb tessellation. (a)Folded flat when γ =180°, (b) deployed into double-curved surface, (c) Lighted form.*

To generate an interesting rotationally symmetric lampshade surface based on a Waterbomb tessellation, the basic unit of a Waterbomb tessellation can be modified. Since the basic Waterbomb unit is rectangular unit, it can be split in the middle so that a Miura strip with divots can be inserted (Figure 24). To make sure the final tessellation is still flat foldable, Kawasaki's Theorem and other flat foldable conditions discussed in Section 4.0 must be satisfied. This altered based unit can then be arrayed to create a new tessellation. Similar to a regular Waterbomb tessellation, this altered tessellation can be rotationally stretched to create lampshades, as shown the Figure 25. The alternation to the original Waterbomb tessellation results in a folded surface that is more suitable for the interplay between light and shadows, generating dramatic effects of gradations of illumination.



(a) (b) (c)

Figure 24. *Adding Miura-ori with divots to a Waterbomb unit: (a) A single Waterbomb unit, (b)A single Waterbomb unit that is split and that is inserted with two Miura-ori with divots strips, (c) Final crease pattern.*

Figure 25. *Two rotationally symmetric lampshades based on the altered Waterbomb tessellation.*

11.0 Conclusion

Though the art of origami itself has a long and intertwined relationship with mathematics, only recently have the mathematics of origami lanterns and lamps been explored. Besides the Miura ori, Yoshimura, and Waterbomb tessellations, other origami tessellations can be mathematically manipulated to create functional and collapsible lampshades (Wu 2018). While the technical aspect of analysing the underlying mathematics in origami design allows us to model the three-dimensional geometry found in a lampshade accurately in mathematics, a great lampshade design is often the result of a combination of craftsmanship, artist vision, and intuition, as well as a mathematical understanding. Many contemporary artists and designers have been particularly drawn to the art of lampshades for multiple reasons. Firstly, the relation of light and shadow, which has always been considered to be among the most fascinating element in all visual arts, has attracted artists and designers alike. Secondly, lighting design is highly influenced by the development of technology, from the first gas lamps and the first electric light bulb to the digital LEDs of today. Today, light and lampshade designers are working with various new trends, including wireless lighting, the internet of things, mood and health lighting, and revolutionary materials using origami designs in order to bring to us lighting experiences that combine functionality and aesthetic beauty.

Acknowledgements:

The background information on paper lanterns and origami in this chapter is developed from my earlier work in Jiangmei Wu (2018): Folding Helical Triangle Tessellations into Light Art, *Journal of the Mathematics and Arts*, Volume 12, Issue 1, pp. 19-33.

I would like to thank Ronald Day and Chunfeng Huang for their useful discussions and suggestions.

References:

Choma, J. 2018. "Foldable Composites for Architectural Applications." Proceedings of the 7th International Meeting on Origami in Science, Mathematics and Education (7OSME).

en.wikipedia.org. 2018. "Gas Lighting." Wikimedia. https://en.wikipedia.org/wiki/Gas_lighting.

Fujimoto, Shuzo. 1976. *Rittai Origami (3D Origami)*. Osaka, Japna: Shinzo Kitagana.

García, Héctor. 2010. *A geek in Japan : [discovering the land of manga, anime, Zen, and the tea ceremony]*. 1st English-language ed. Tokyo ; Rutland, Vt.: Tuttle Pub.

Gurkewitz, Rona, and Bennett Arnstein. 2003. *Multimodular origami polyhedra : archimedeans, buckyballs, and duality*. Mineola, N.Y.: Dover Publications.

Huffman, D.A. 1976. "Curvature and creases: A primer on paper." *IEEE Transactions on Computers* C-25 (10):1010-1019.

Hughes, S. 1978. *Washi: The world of Japanese paper*. Tokyo and New York: Kodansha International, Ltd., through Harper & Row, Ltd.

Hull, Thomas. 1997. "Five Intersecting Tetrahedra." http://mars.wne.edu/~thull/fit.html.

Hull, Thomas. 2011. "Solving Cubics With Creases: The Work of Beloch and Lill." *American Mathematical Monthly* 118:307-315. doi: 10.4169/amer.math.monthly.118.04.307.

Huzita, H. 1989. "The trisection of a given angle solved by the geometry of origami." First International Meeting of Origami Science and Technology, Ferara, Italy, December 1989.

Kasahara, Kunihiko. 2004. *The Art and Wonder of Origami*. Massachusetts: Quarry Books.

Kida, Takuya. 2003. *Akari: Light Sculpture by Isamu Noguchi*. Tokyo: National Museum of Modern Art.

Klint, Le. 2018. "Unfolding 75 Years of Lighting." https://www.leklint.com/en-GB/News/Unfolding-75-years-of-lighting.aspx?Action=1&PID=820.

Kuribayashi, K., K. Tsuchiya, Z. You, D. Tomus, M. Umemoto, T. Ito, and M. Sasaki. 2006. "Self-Deployable Origami Stent Grafts as a Biomedical Application of Ni-rich TiNi Shape Membory Alloy Foil." *Mater, Sci. and Engr.* A419:131-137.

Lang, Robert J. 1996. "Origami and Geometric Constructions." http://www.langorigami.com/files/articles/origami_constructions.pdf.

Lang, Robert J. 2012. *Origami design secrets : mathematical methods for an ancient art*. 2nd ed. Boca Raton: A K Peters/CRC Press.
Lang, Robert J. 2018. *Twists, tilings, and tessellations : mathematical methods for geometric origami*. Boca Raton: CRC Press, Taylor & Francis Group.
Mitani, Jun. 2011. "A Method for Designing Crease Patterns for Flat-Foldable Origami with Numerical Optimization." *Journal for Geometry and Graphics* 15 (2):195-201.
Miura, K. 2009. "The Science of Miura-Ori: A Review." Fourth International Meeting of Origami: Science, Mathematics, and Education.
Morriseey, J. 2019. "Art by the Numbers." https://www.nytimes.com/2019/10/23/arts/momath-origami.html.
Nichols, K., J. Elgar, and K. Gausch. 2007. "Illuminating the Way: Conservation of Two Japanese Paper Lanterns." *Journal of the American Institute for Conservation* 46 (2):123-136.
Ortega, Pedro Ribera. 1973. *Christmas in old Santa Fe*. 2d ed. Santa Fe, N.M.,: Sunstone Press.
pbs.org. 2017. "Between the Folds: History of Origami." https://www.pbs.org/independentlens/between-the-folds/history.html.
Quin, Scala, and Fletcher Sibthorp. 2012. *Lighting: 20th Century Classics*. London, Captown, Sydney, Auckland: New Holland Publisher.
Raison, Laura. 1994. *Florence & Tuscany*, *Passport's regional guides of Italy*. Lincolnwood, Ill., USA: Passport Books.
Resch, Ronald D. 1973. "The topological design of sculptural and architectural systems." National Computer Conference and Exposition New York, NY, June 4-8, 1973.
Resch, Ronald D. 1968. Self- Supporting Structural Unit Having a Series of Repetitious Geometric Modules. edited by US Patent.
Rutzky, Jeffrey, and Chris K. Palmer. 2011. *Shadowfolds : surprisingly easy-to-make geometric designs in fabric*. New York: Kodansha International.
Sallas, Joan. 2010. *Gefaltete Schönheit*. Freiburg im Breisgau un Wien: Privately published.
Shumakov, Yuri, and Katrin Shumakav. 2015. "Origami Magic Ball Wonders: From Dragon's Egg to Hot Air Ballons." www.oriland.com.
Song, L. 2015. "The Lantern Festival." In: Paths International Ltd. https://search-ebscohost-com.proxyiub.uits.iu.edu/login.aspx?direct=true&db=cat00016a&AN=inun.17309445&site=eds-live&scope=site.
Sundara Rao, T. 1901. *T. Sundara Row's Geometric exercises in paper folding*. Chicago,: The Open court pub. company; etc.
Wingler, Hans Maria, and Joseph Stein. 1976. *The Bauhaus : Weimar, Dessau, Berlin, Chicago*. Cambridge, Massachuestts: The MIT Press. still image.
Wu, J. 2017. "Body, Form, Material and Surface: Making of Ruga Interior Skin." *Interiors: Design/Architecture/Culture* 8 (3):73-87.
Wu, J. 2018. "Folding Helical Triangle Tessellation into Light Art." *Journal of Mathematics and Arts* 12 (1):19-33.
Wu, J., S. Ressel, and K. Overton. 2018. "Light Harvest: Interactive Sculptural Installation based on Folding and Mapping Proteins." *Digital Creativity* 29 (4):274-286.
www.isseymiyake.com. 2017. "In-Ei Issey Miyake." accessed August 1st. https://www.isseymiyake.com/en/brands/in-ei.

Yoshimura, Y. 1955. "On the Mechanism of a Circular Cylindrical Shell under Axial Compression." *Technical report, NACA TM 1390*.